Fundamentals of Accelerator 2012

Day 3

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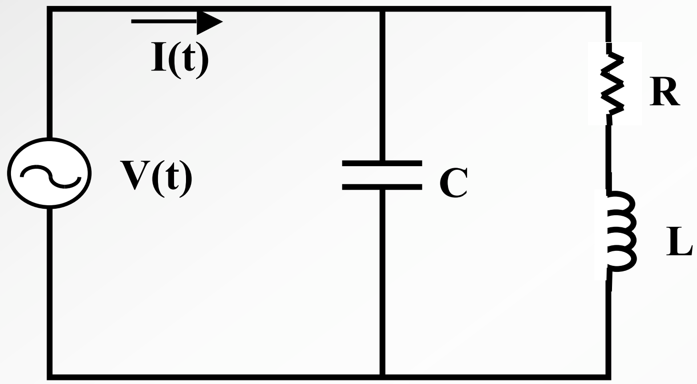
Lumped circuit analogy of resonant cavity



$$Z(\omega) = \left[j\omega C + (j\omega L + R)^{-1} \right]^{-1}$$

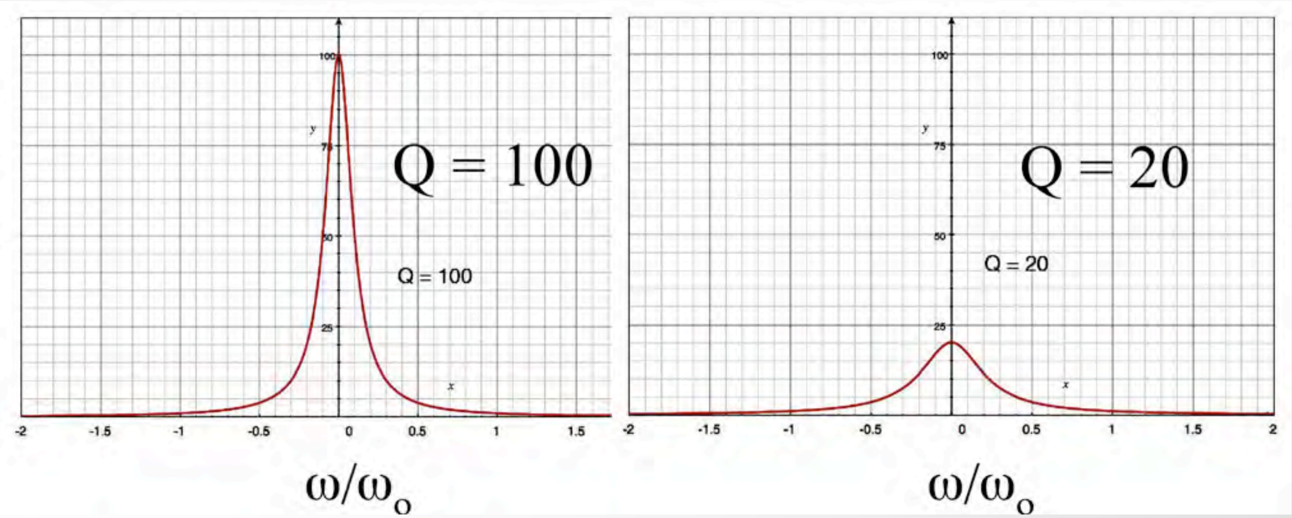
$$Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R)\, j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC}$$

The resonant frequency is $\omega_o = 1/\sqrt{LC}$

Q of the lumped circuit analogy

Converting the denominator of Z to a real number we see that

$$|Z(\omega)| \sim \left[\left(1-\frac{\omega^2}{\omega_o^2}\right)^2 + (\omega RC)^2\right]^{-1}$$



The width is $\frac{\Delta\omega}{\omega_0} = \frac{R}{\sqrt{L/.C}}$

More basics from circuits - Q

$$Q = \frac{\omega_o \circ \; Energy\ stored}{Time\ average\ power\ loss} = \frac{2\pi \circ Energy\ stored}{Energy\ per\ cycle}$$

$$\mathscr{E} = \frac{1}{2} L\, I_o I_o^* \qquad \text{and } \langle \mathscr{P} \rangle = \langle \mathrm{i}^2(t) \rangle R = \frac{1}{2} I_o I_o^* R_{surface}$$

$$\therefore \; Q = \frac{\sqrt{L/C}}{R} = \left(\frac{\Delta\omega}{\omega_o} \right)^{-1}$$

Translate circuit model to a cavity model:
Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

$$L = \frac{\mu_o \pi a^2}{2\pi(R+a)}$$

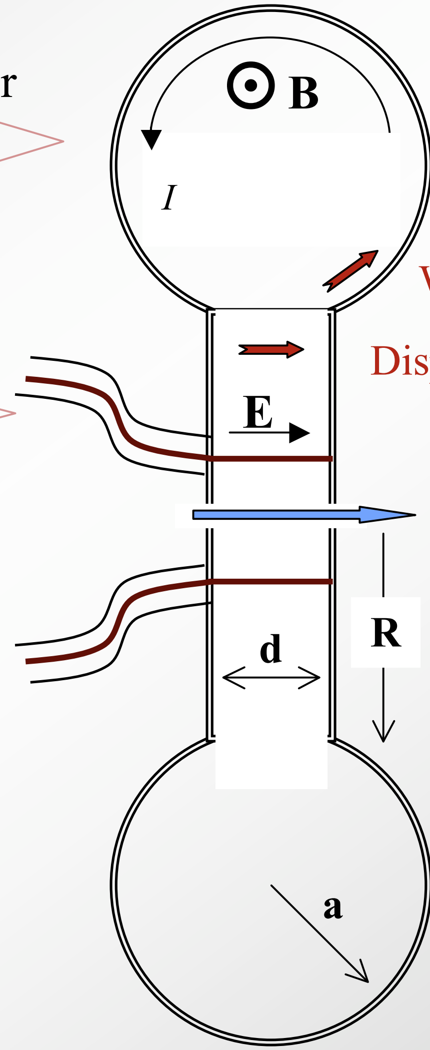
Central region: Large plate Capacitor

$$C = \varepsilon_o \frac{\pi R^2}{d}$$

$$\omega_o = 1/\sqrt{LC} = c\left[\frac{2((R+a)d}{\pi R^2 a^2}\right]^{1/2}$$

Q – set by resistance in outer region

$$Q = \sqrt{L/C}/R$$

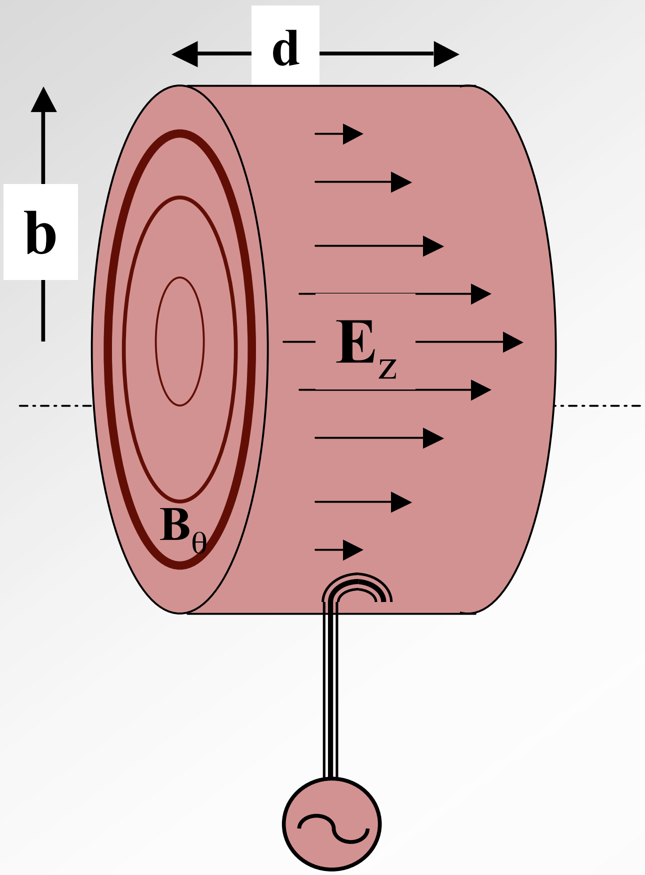


Expanding outer region raises Q

Narrowing gap raises shunt impedance

Source: Humphries, Charged Particle Accelerators

Properties of the RF pillbox cavity



$$\sigma_{walls} = \infty$$

- We want lowest mode: with only $\mathbf{E}_z$ & $\mathbf{B}_\theta$
- Maxwell's equations are:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \frac{1}{c^2}\frac{\partial}{\partial t}E_z \quad \text{and} \quad \frac{\partial}{\partial r}E_z = \frac{\partial}{\partial t}B_\theta$$

- Take derivatives

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right)\right] = \frac{\partial}{\partial t}\left[\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\right] = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial}{\partial r}\frac{\partial E_z}{\partial r} = \frac{\partial}{\partial r}\frac{\partial B_\theta}{\partial t}$$

==>

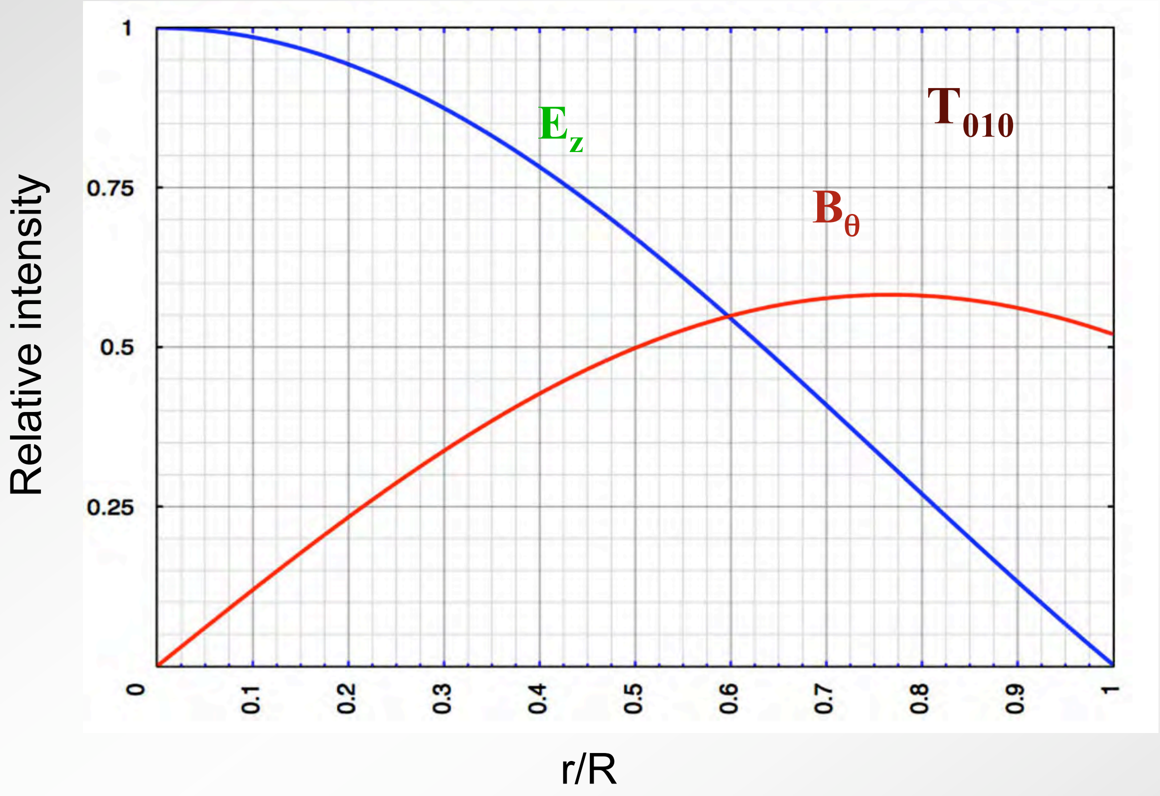
$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

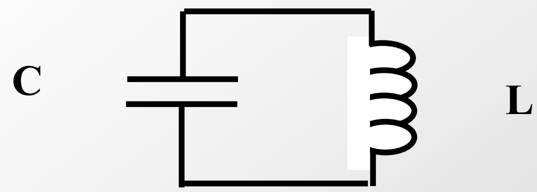
For a mode with frequency ω

- $$E_z(r,t) = E_z(r)\, e^{i\omega t}$$
- Therefore, $$E_z'' + \frac{E_z'}{r} + \left(\frac{\omega}{c}\right)^2 E_z = 0$$
  - (Bessel's equation, 0 order)
- Hence, $$E_z(r) = E_o\, J_o\left(\frac{\omega}{c} r\right)$$
- For conducting walls, $E_z(R) = 0$, therefore

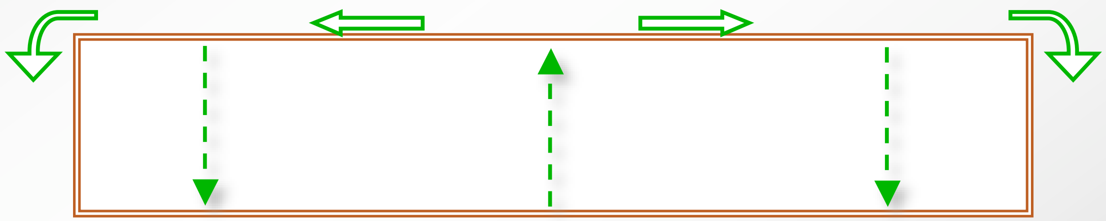
$$\frac{2\pi f}{c} b = 2.405$$

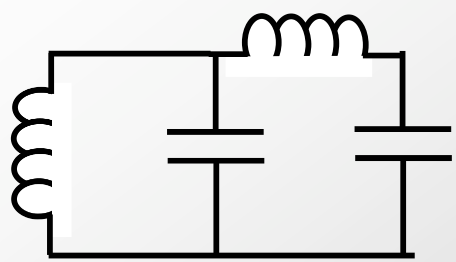
E-fields & equivalent circuit: T_{on1o} mode



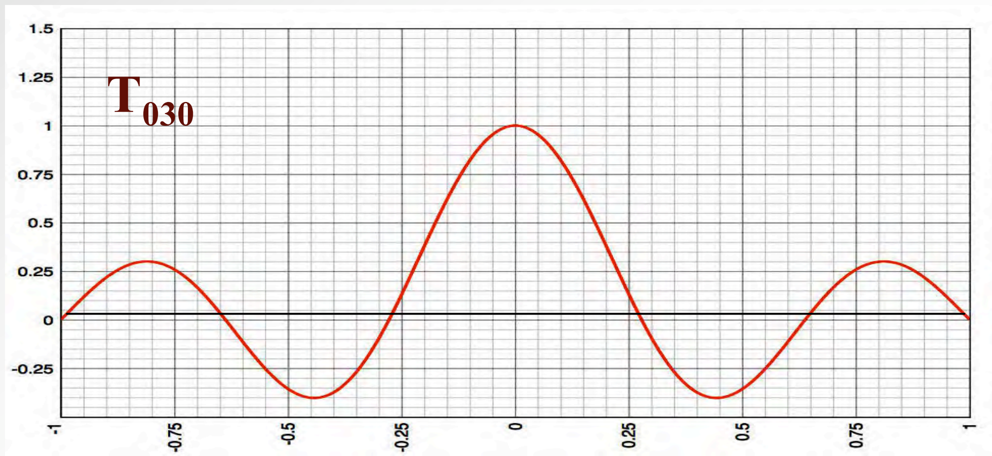


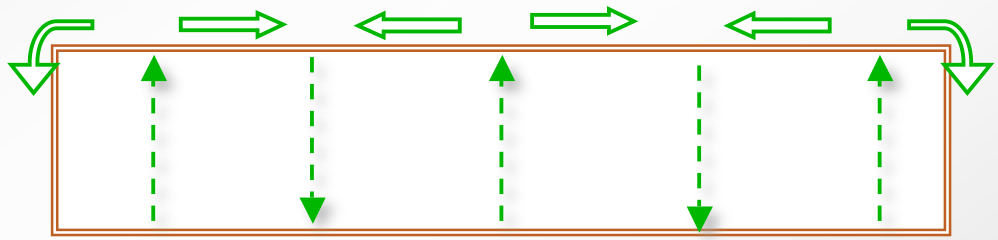
E-fields & equivalent circuits for T_{o2o} modes

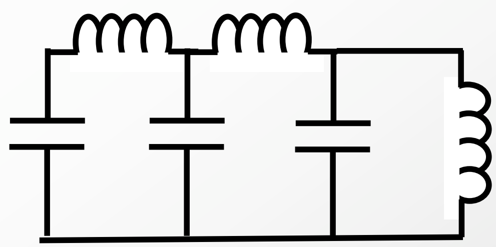




E-fields & equivalent circuits for T_{ono} modes

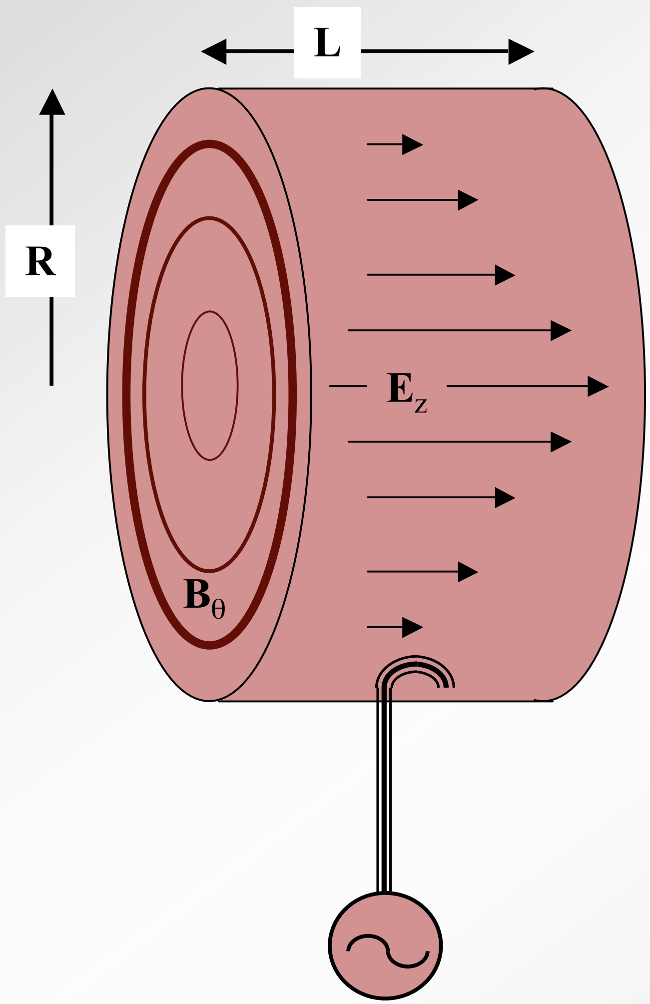






T_{0n0} has n coupled, resonant circuits; each L & C reduced by 1/n

Simple consequences of pillbox model



- Increasing R lowers frequency

  ==> Stored Energy, $\mathcal{E} \sim \omega^{-2}$

- $\mathcal{E} \sim E_z^2$

- Beam loading lowers E_z for the next bunch

- Lowering ω lowers the fractional beam loading

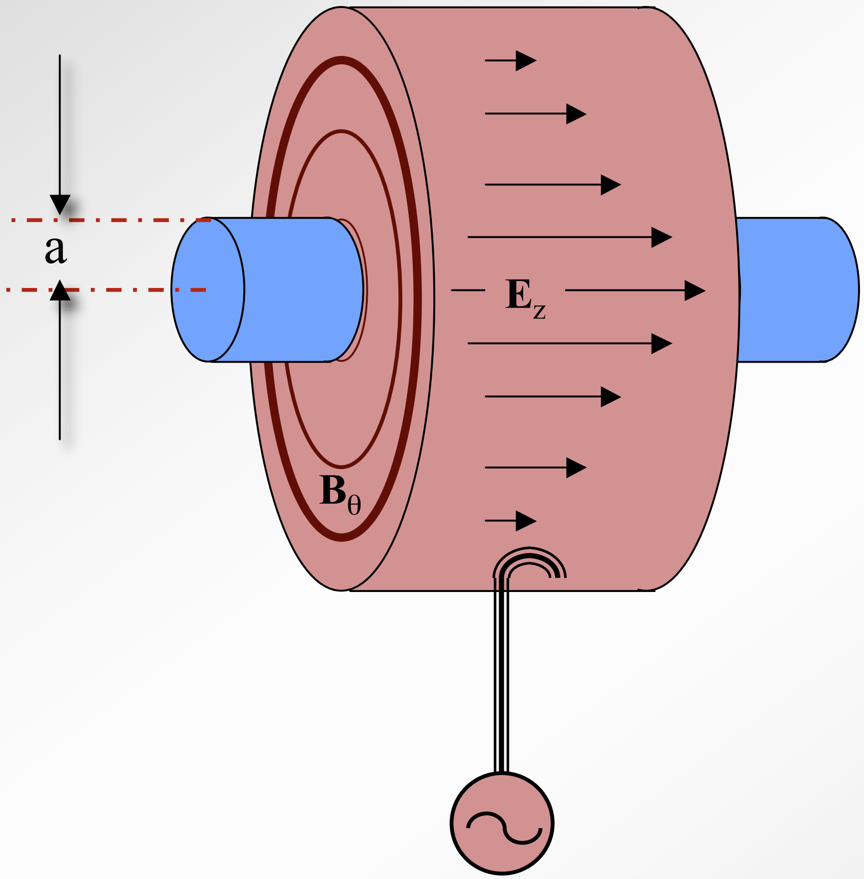
- Raising ω lowers $Q \sim \omega^{-1/2}$

- If time between beam pulses,

  $T_s \sim Q/\omega$

  almost all $\mathcal{E}$ is lost in the walls

The beam tube makes field modes (& cell design) more complicated



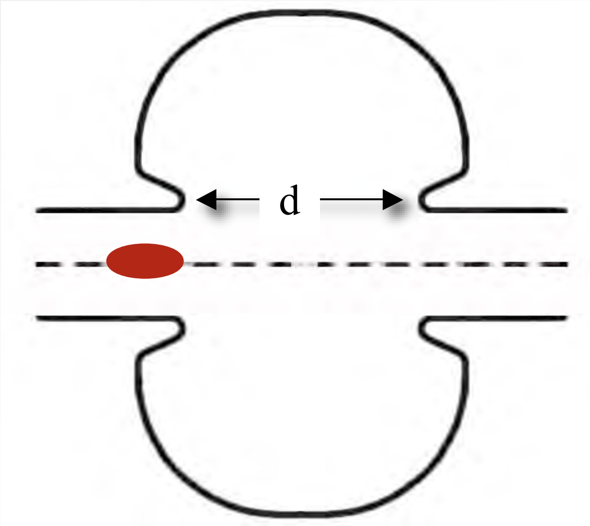
- Peak E no longer on axis
  - $E_{pk} \sim 2 - 3 \times E_{acc}$
  - $FOM = E_{pk}/E_{acc}$
- ω_o more sensitive to cavity dimensions
  - Mechanical tuning & detuning
- Beam tubes add length & €'s w/o acceleration
- Beam induced voltages $\sim a^{-3}$
  - Instabilities

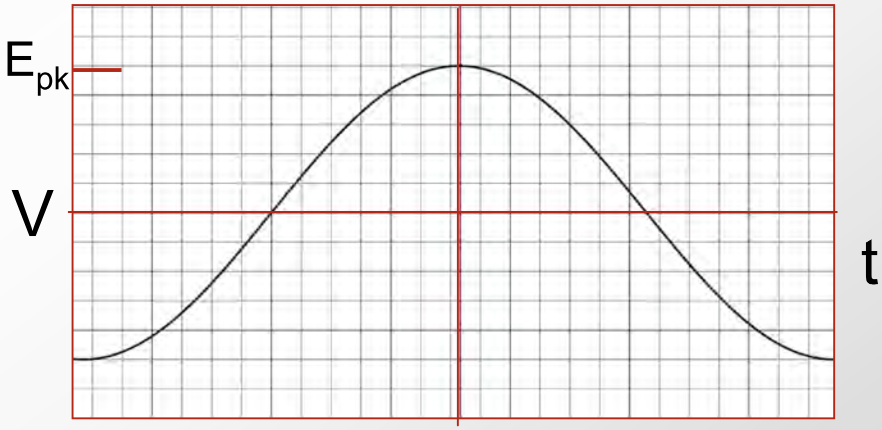
Cavity figures of merit

Figure of Merit: Accelerating voltage

- The voltage varies during time that bunch takes to cross gap
  - reduction of the peak voltage by Γ (transt time factor)

$$\Gamma = \frac{\sin\left(\vartheta/2\right)}{\vartheta/2} \quad \text{where} \quad \vartheta = \omega d/\beta c$$





For maximum acceleration $T_{\mathrm{cav}} = \frac{d}{c} = \frac{T_{\mathrm{rf}}}{2}$ ==> $\Gamma = 2/\pi$

Figure of merit from circuits - Q

$$Q = \frac{\omega_o \circ \ Energy\ stored}{Time\ average\ power\ loss} = \frac{2\pi \circ Energy\ stored}{Energy\ lost\ per\ cycle}$$

$$\mathscr{E} = \frac{\mu_o}{2} \int_v |H|^2 dv = \frac{1}{2} L\, I_o I_o^*$$

$$\langle \mathscr{P} \rangle = \frac{R_{surf}}{2} \int_s |H|^2 ds = \frac{1}{2} I_o I_o^* R_{surf}$$

$$R_{surf} = \frac{1}{Conductivity \circ Skin\ depth} \sim \omega^{1/2}$$

$$\therefore \quad Q = \frac{\sqrt{L/C}}{R_{surf}} = \left(\frac{\Delta\omega}{\omega_o} \right)^{-1}$$

Measuring the energy stored in the cavity allows us to measure Q

- We have computed the field in the fundamental mode

$$U = \int_0^d dz \int_0^b dr 2\pi r \left(\varepsilon E_o^2 / 2 \right) J_1^2(2.405 r/b)$$

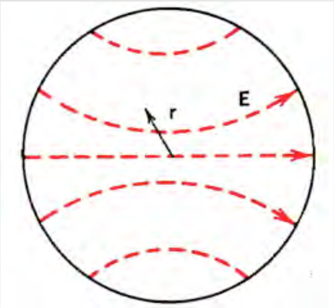
$$= b^2 d \left(\varepsilon E_o^2 / 2 \right) J_1^2(2.405)$$

- To measure Q we excite the cavity and measure the E field as a function of time
- Energy lost per half cycle = UπQ
- Note: energy can be stored in the higher order modes that deflect the beam

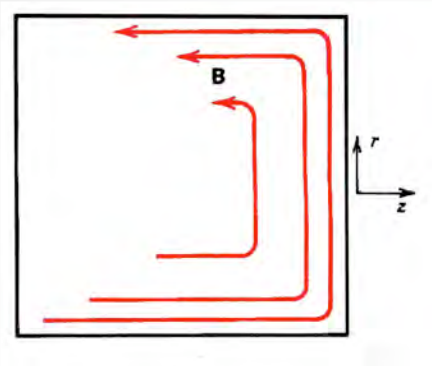
Keeping energy out of higher order modes

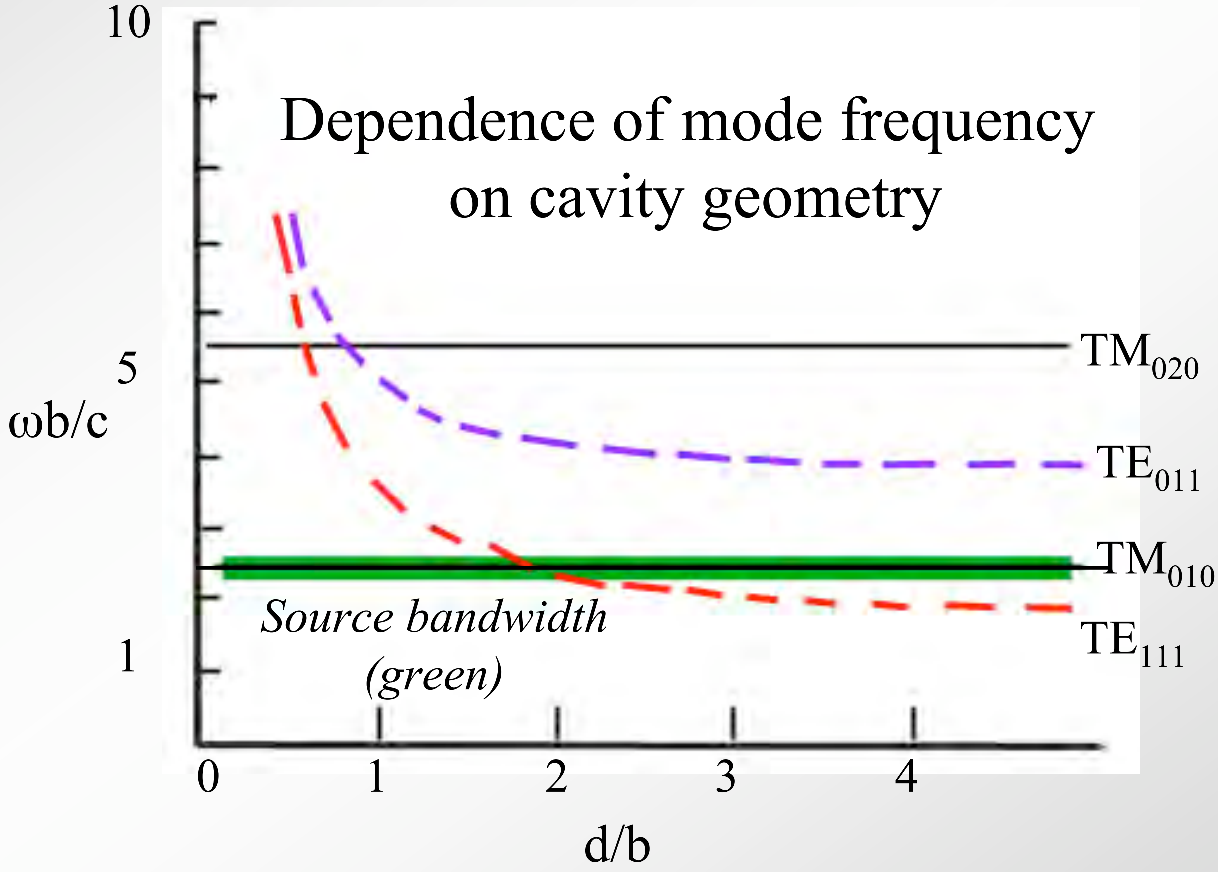
TE$_{111}$ mode

End view



Side view





Choose cavity dimensions to stay far from crossovers

Figure of merit for accelerating cavity:
Power to produce the accelerating field

Resistive input (shunt) impedance at ω_o relates power dissipated in walls to accelerating voltage

$$R_{in} = \frac{\langle V^2(t) \rangle}{\mathscr{P}} = \frac{V_o^2}{2\mathscr{P}} = Q\sqrt{L/C}$$

Linac literature commonly defines "shunt impedance" without the "2"

$$\mathscr{R}_{in} = \frac{V_o^2}{\mathscr{P}} \sim \frac{1}{R_{surf}}$$

Typical values 25 - 50 MΩ

Computing shunt impedance

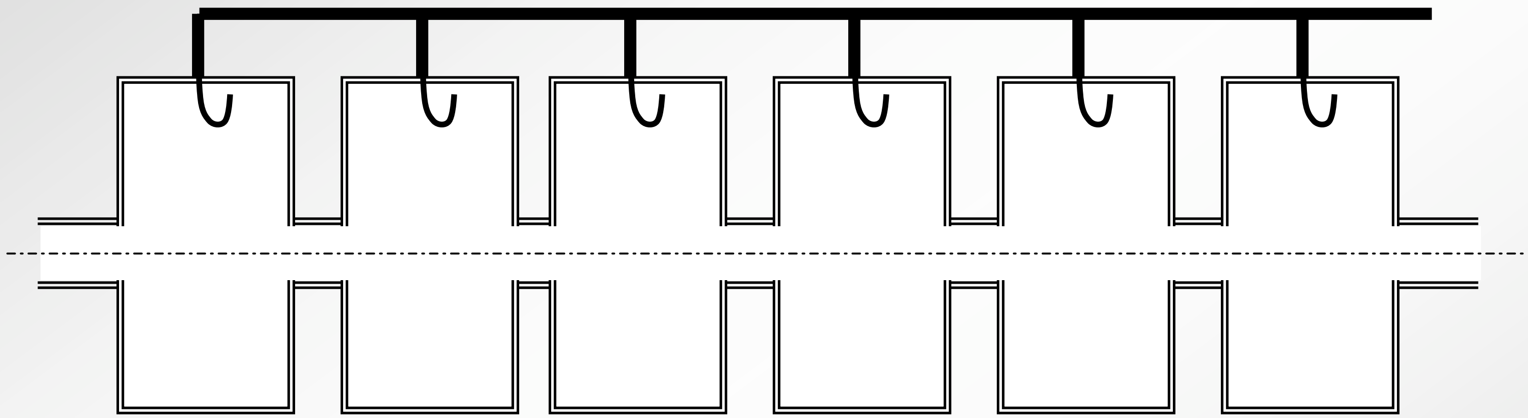
$$\mathscr{R}_{in} = \frac{V_o^2}{\mathscr{P}}$$

$$\langle \mathscr{P} \rangle = \frac{R_{surf}}{2} \int_s |H|^2 ds$$

$$R_{surf} = \frac{\mu\omega}{2\sigma_{dc}} = \pi Z_o \frac{\delta_{skin}}{\lambda_{rf}} \quad \text{where } Z_o = \sqrt{\mu_o / \varepsilon_o} = 377\Omega$$

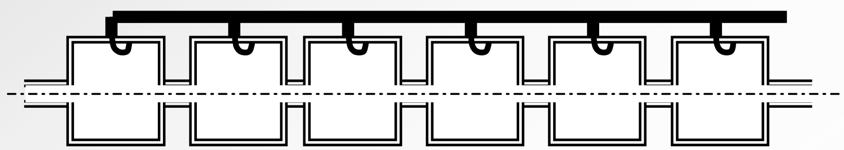
The on-axis field E and surface H are generally computed with a computer code such as SUPERFISH for a complicated cavity shape

To make a linac
Use a series of pillbox cavities

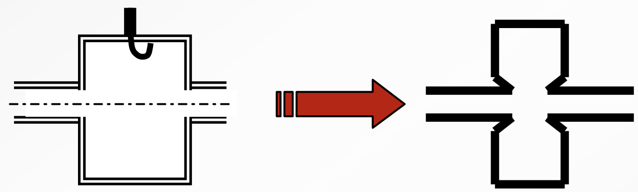


Power the cavities so that $E_z(z,t) = E_z(z)e^{i\omega t}$

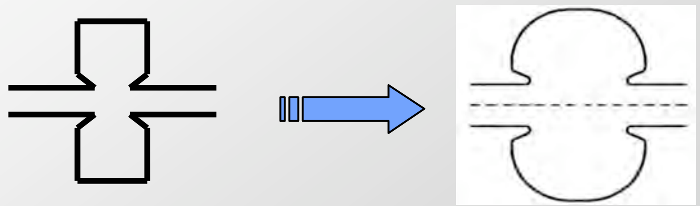
Improve on the array of pillboxes?



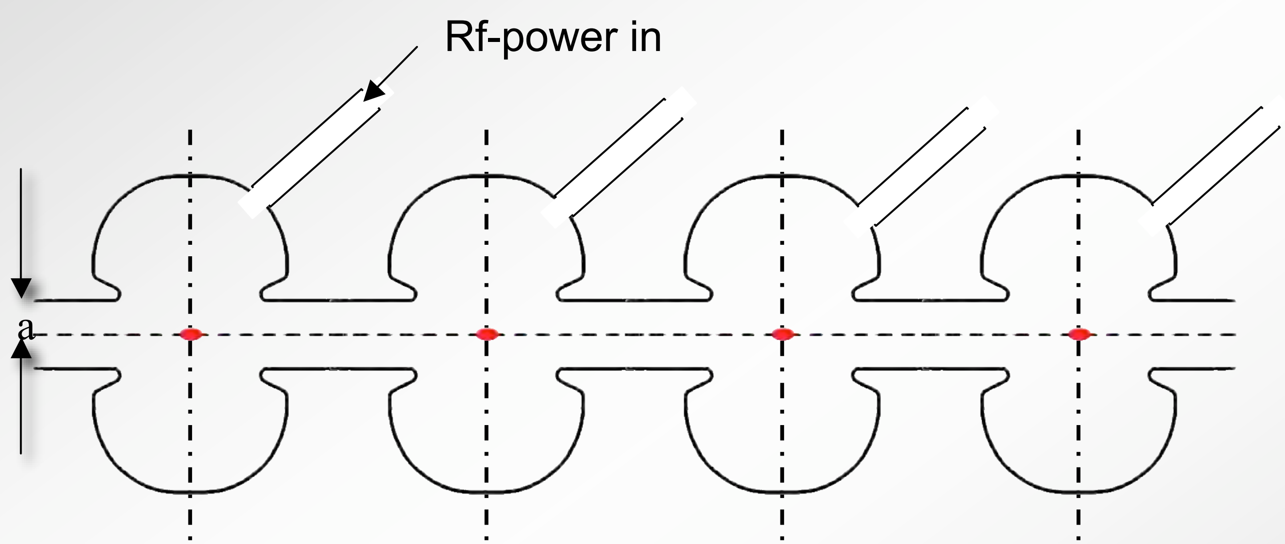
- Return to the picture of the re-entrant cavity



- Nose cones concentrate E_z near beam for fixed stored energy
- Optimize nose cone to maximize V^2; I.e., maximize R_{sh}/Q
- Make H-field region nearly spherical; raises Q & minimizes P for given stored energy



Thus, linacs can be considered to be an array of distorted pillbox cavities…

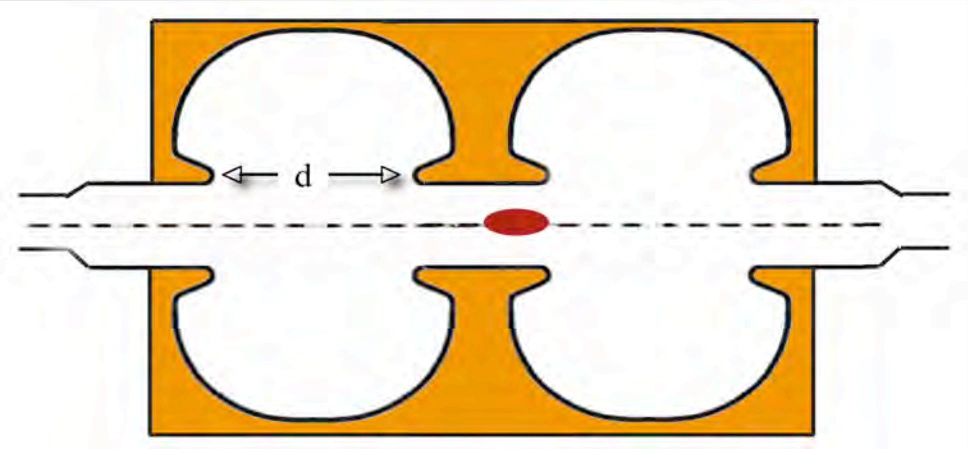


In warm linacs "nose cones" optimize the voltage per cell with respect to resistive dissipation

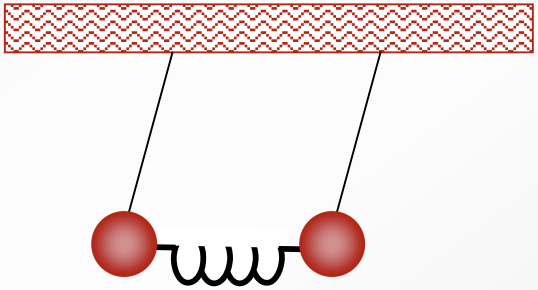
$$Q = \frac{\sqrt{L/C}}{\mathscr{R}_{surface}}$$

Usually cells are feed in groups not individually…. and

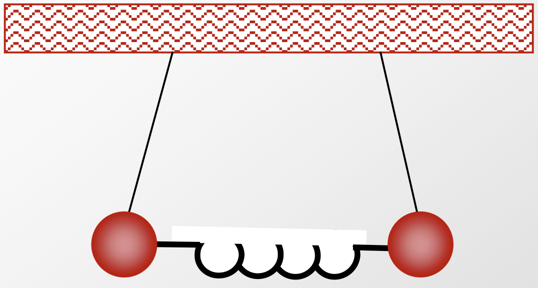
Linacs cells are linked to minimize cost



==> coupled oscillators ==>multiple modes

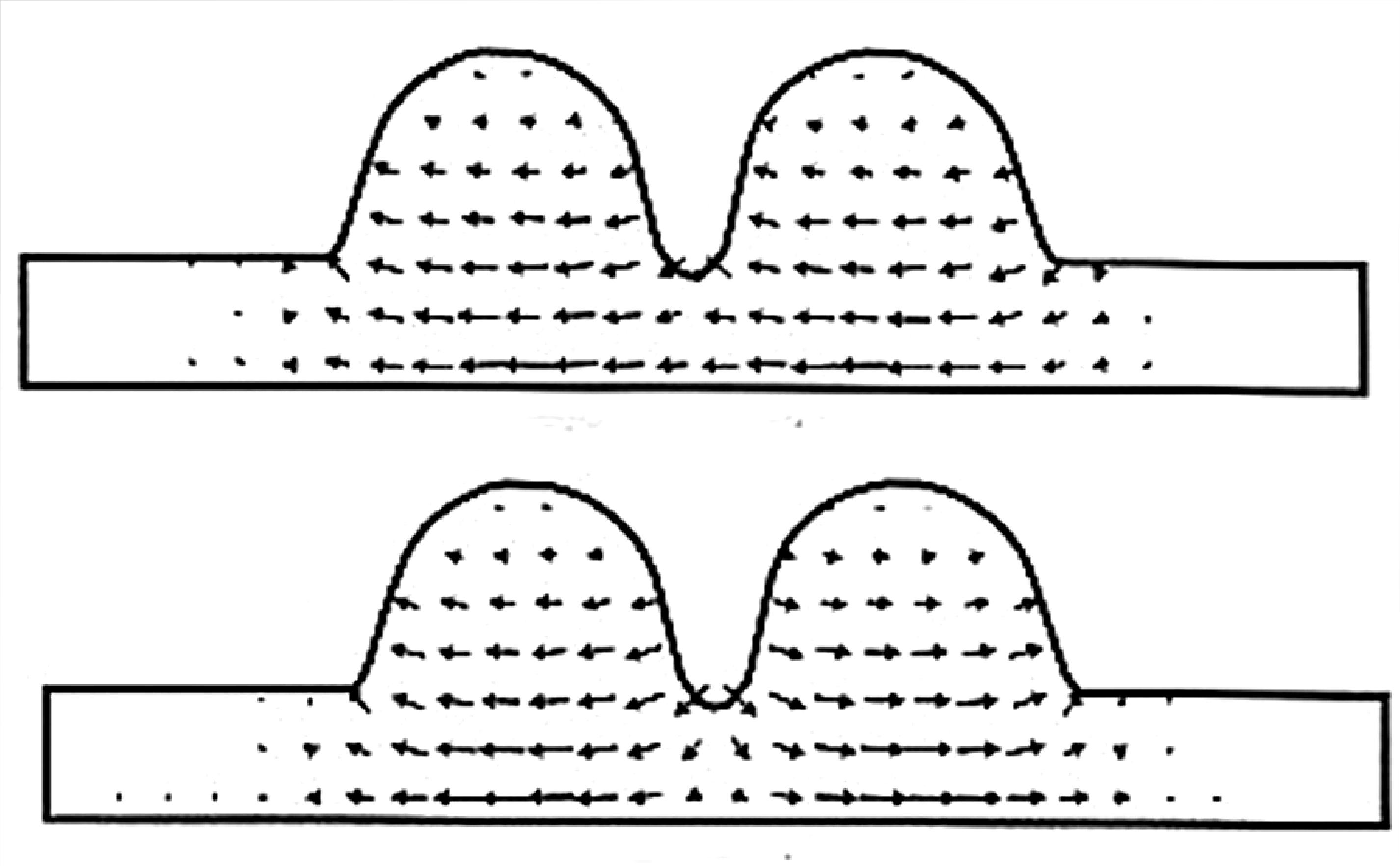


Zero mode

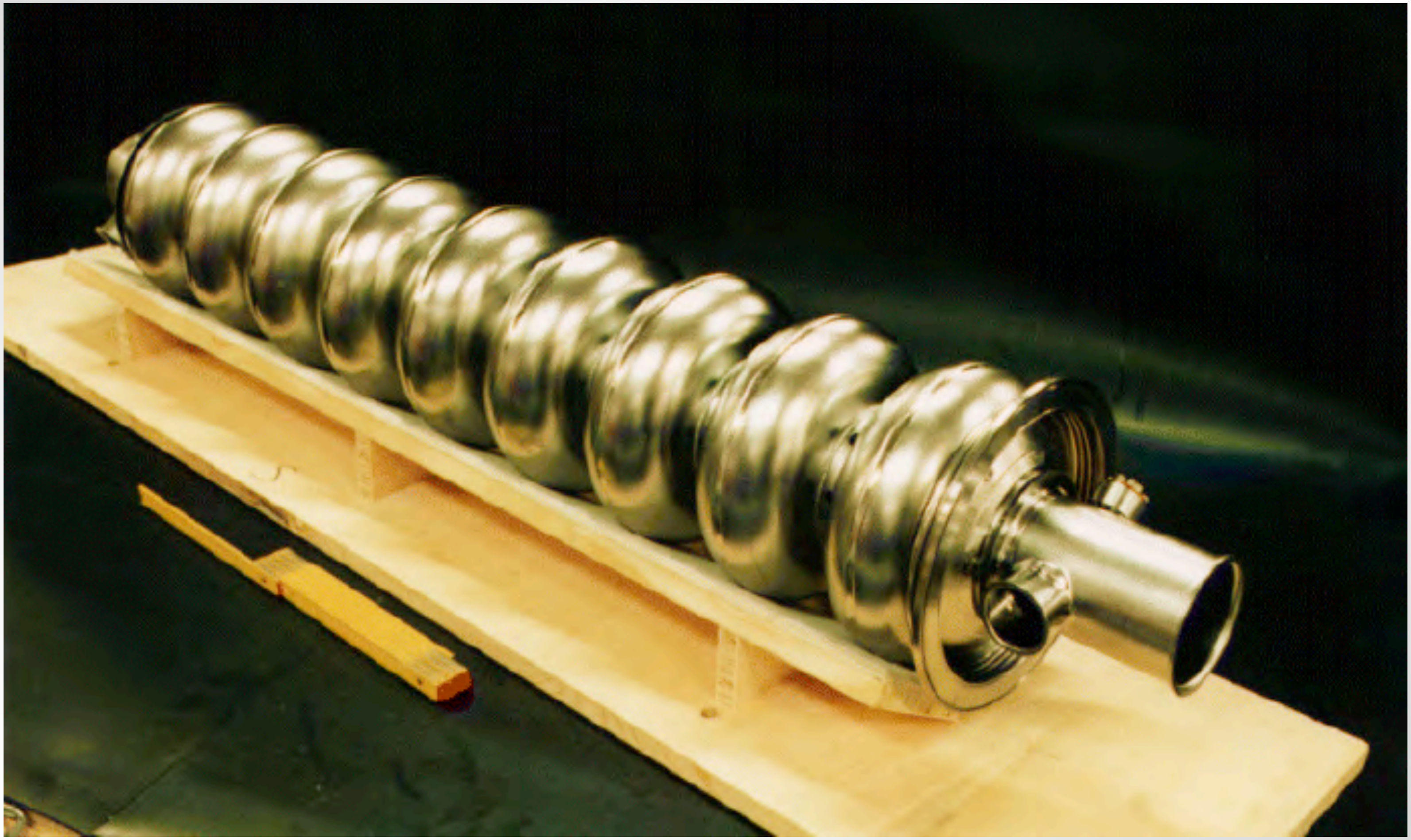


π mode

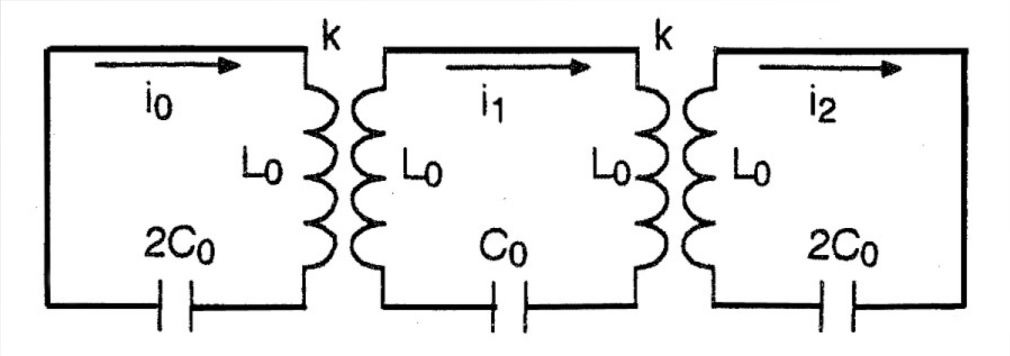
Modes of a two-cell cavity



9-cavity TESLA cell



Example of 3 coupled cavities



$$x_0\left(1-\frac{\omega_0^2}{\Omega^2}\right)+x_1 k=0 \qquad \text{oscillator } n=0$$

$$x_1\left(1-\frac{\omega_0^2}{\Omega^2}\right)+(x_0+x_2)\frac{k}{2}=0 \qquad \text{oscillator } n=1$$

$$x_2\left(1-\frac{\omega_0^2}{\Omega^2}\right)+x_1 k=0 \qquad \text{oscillator } n=2$$

$x_j = i_j\sqrt{2L_o}$ and Ω = normal mode frequency

Write the coupled circuit equations in matrix form

$$\mathbf{L}\mathbf{x}_q = \frac{1}{\Omega_q^2}\mathbf{x}_q \quad \text{where} \quad \mathbf{L} = \begin{pmatrix} 1/\omega_o^2 & k/\omega_o^2 & 0 \\ k/2\omega_o^2 & 1/\omega_o^2 & k/2\omega_o^2 \\ 0 & k/\omega_o^2 & 1/\omega_o^2 \end{pmatrix} \quad \text{and} \quad \mathbf{x_q} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Compute eigenvalues & eigenvectors to find the three normal modes

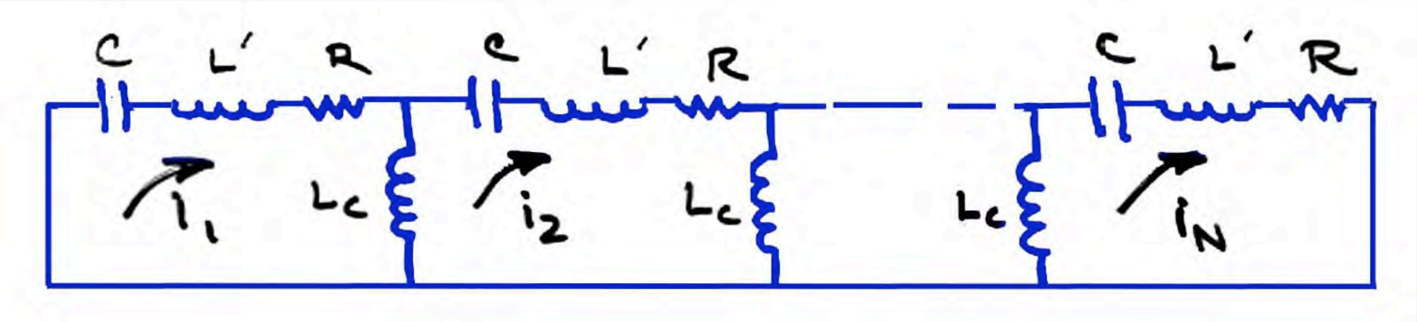
Mode q = 0 : zero mode $\quad \Omega_0 = \dfrac{\omega_o}{\sqrt{1+k}} \quad \mathbf{x_0} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Mode q = 1 : $\pi/2$ mode $\quad \Omega_1 = \omega_o \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Mode q = 2 : π mode $\quad \Omega_2 = \dfrac{\omega_o}{\sqrt{1-k}} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

For a structure with N coupled cavities

- ==> Set of N coupled oscillators
  - N normal modes, N frequencies
- From the equivalent circuit with magnetic coupling

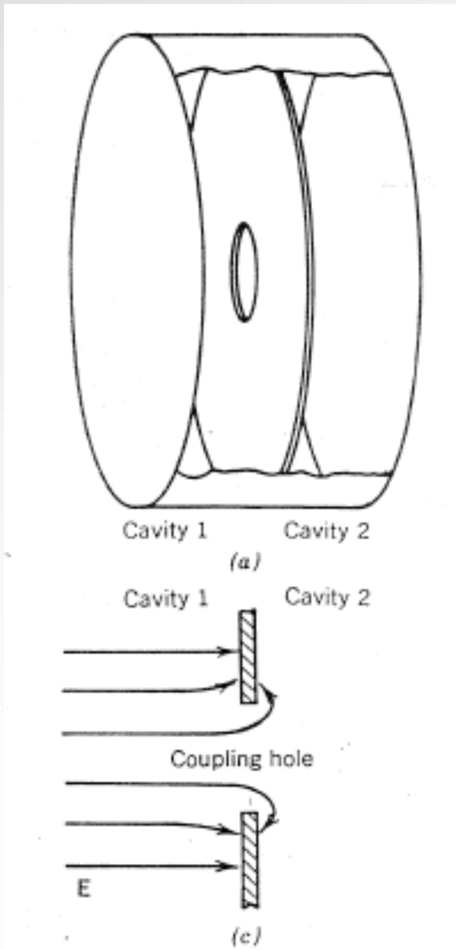


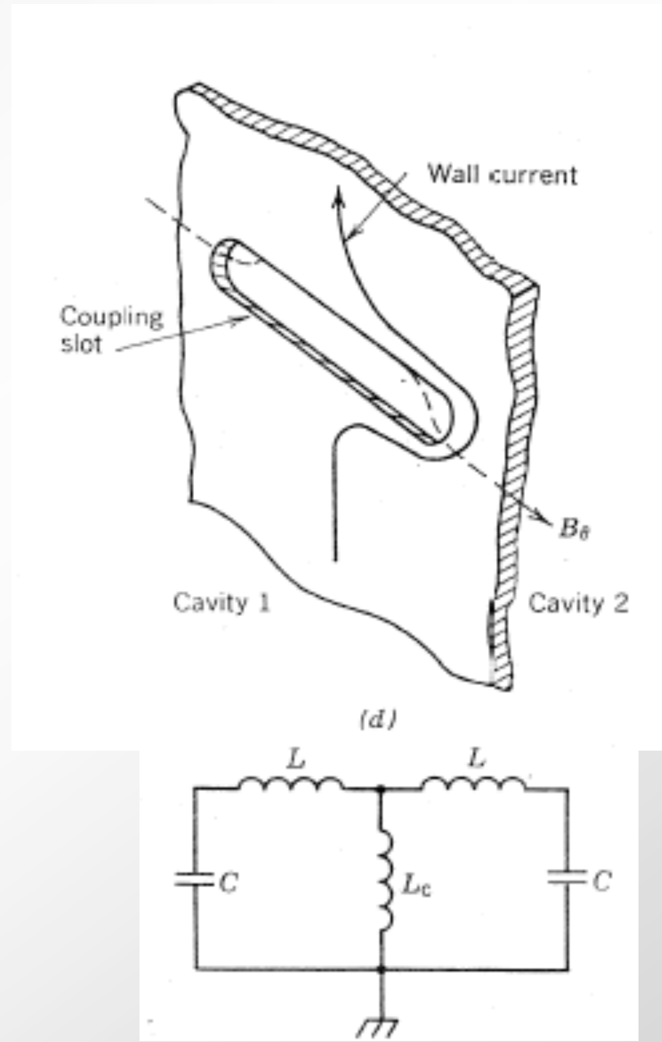
$$\omega_m = \frac{\omega_o}{\left(1 - B\cos\frac{m\pi}{N}\right)^{1/2}} \approx \omega_o\left(1 + B\cos\frac{m\pi}{N}\right)$$

where B = bandwidth (frequency difference between lowest & high frequency mode)

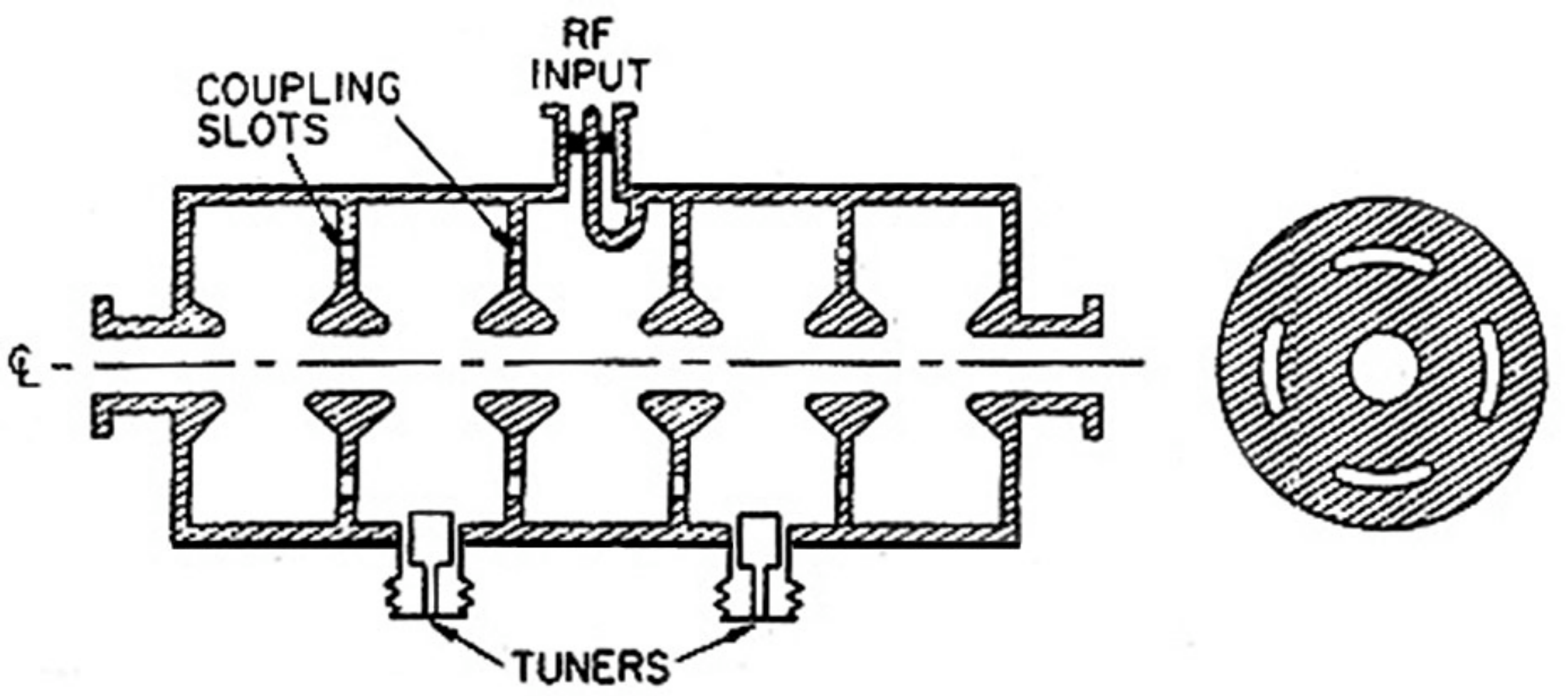
- Typically accelerators run in the π-mode

Magnetically coupled pillbox cavities





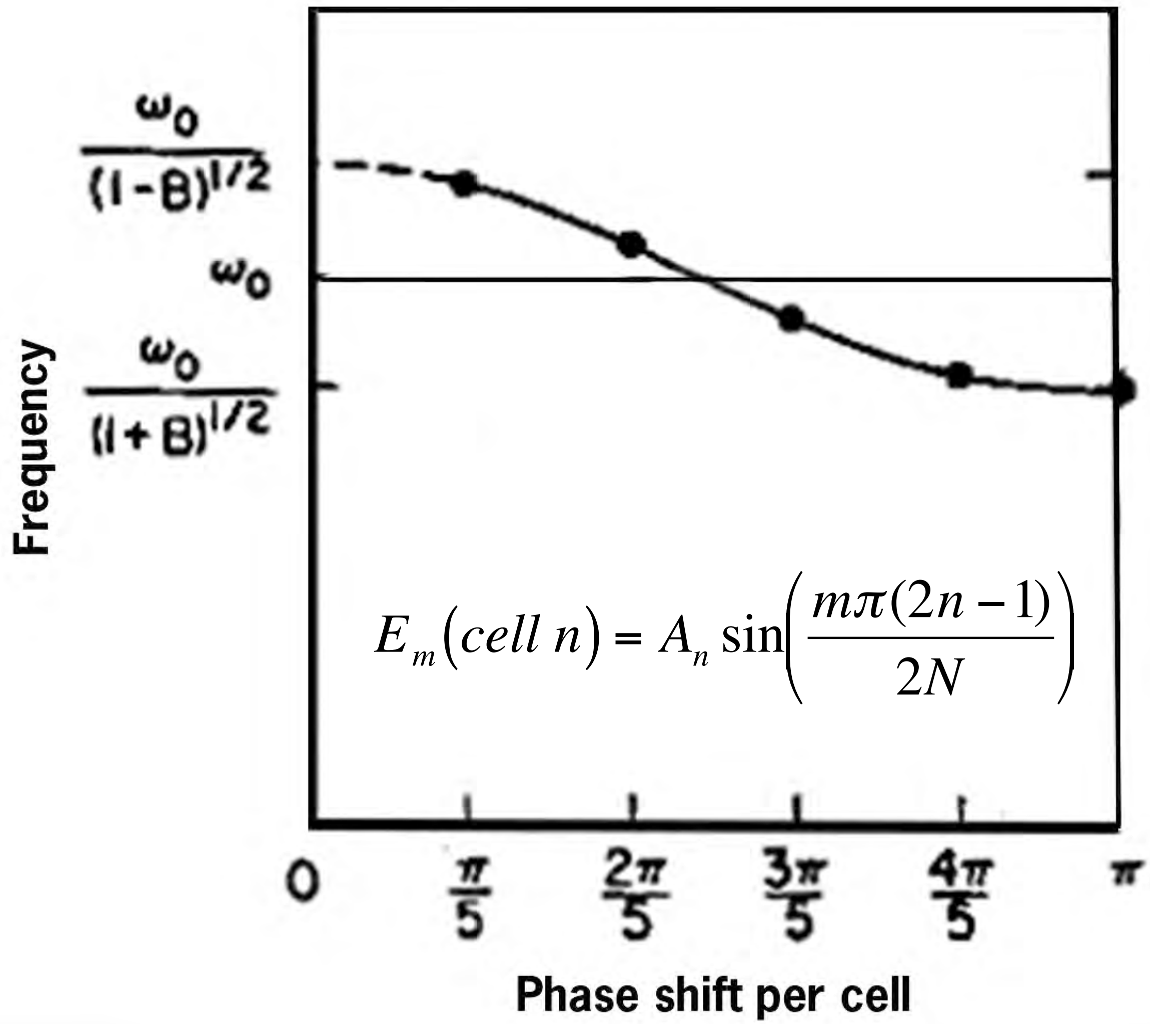
5-cell π-mode cell with magnetic coupling



The tuners change the frequencies by perturbing wall currents ==> changes the inductance ==> changes the energy stored in the magnetic field

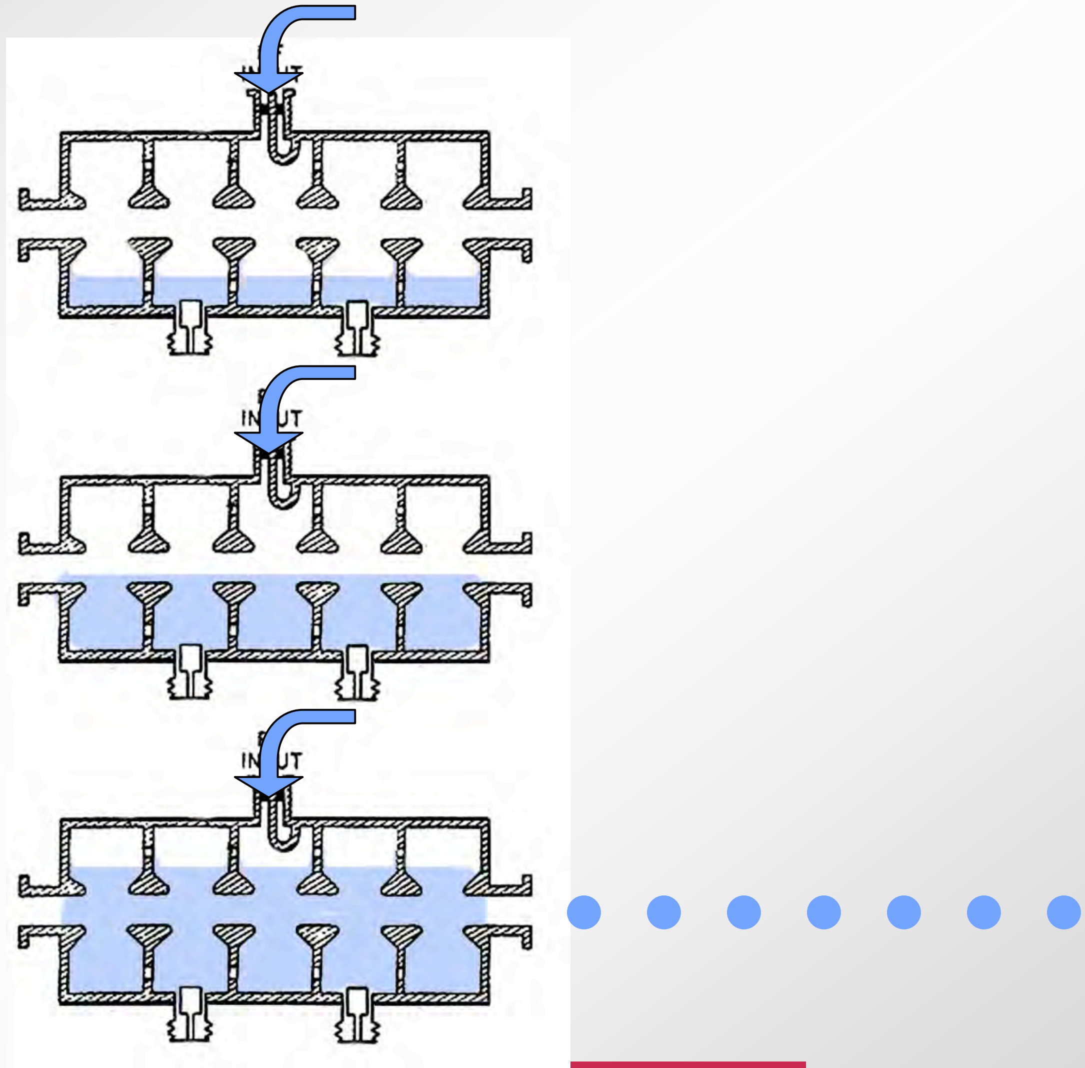
$$\frac{\Delta\omega_o}{\omega_o} = \frac{\Delta U}{U}$$

Dispersion diagram for 5-cell structure



$$E_m(cell\ n) = A_n \sin\left(\frac{m\pi(2n-1)}{2N}\right)$$

Schematic of energy flow in a standing wave structure



What makes SC RF attractive?

Comparison of SC and NC RF

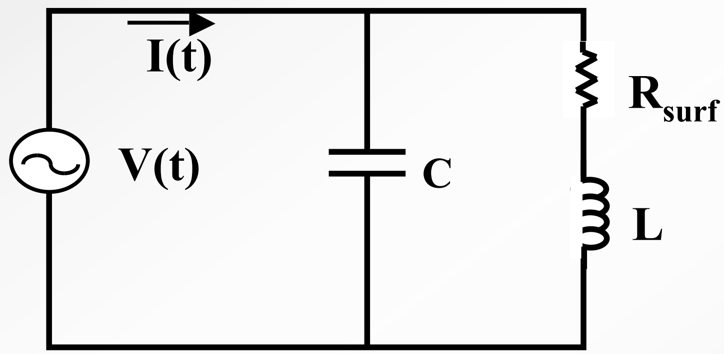
Superconducting RF

- High gradient
  ==> 1 GHz, meticulous care
- Mid-frequencies
  ==> Large stored energy, $\mathscr{E}_s$
- Large $\mathscr{E}_s$
  ==> very small ΔE/E
- Large Q
  ==> high efficiency

Normal Conductivity RF

- High gradient
  ==> high frequency (5 - 17 GHz)
- High frequency
  ==> low stored energy
- Low $\mathscr{E}_s$
  ==> ~10x larger ΔE/E
- Low Q
  ==> reduced efficiency

Recall the circuit analog



As R_{surf} ==> 0, the Q ==> ∞.

In practice,

$Q_{nc} \sim 10^4$ $Q_{sc} \sim 10^{11}$

Figure of merit for accelerating cavity: power to produce the accelerating field

Resistive input (shunt) impedance at ω_o relates power dissipated in walls to accelerating voltage

$$R_{in} = \frac{\langle V^2(t) \rangle}{\mathscr{P}} = \frac{V_o^2}{2\mathscr{P}} = Q\sqrt{L/C}$$

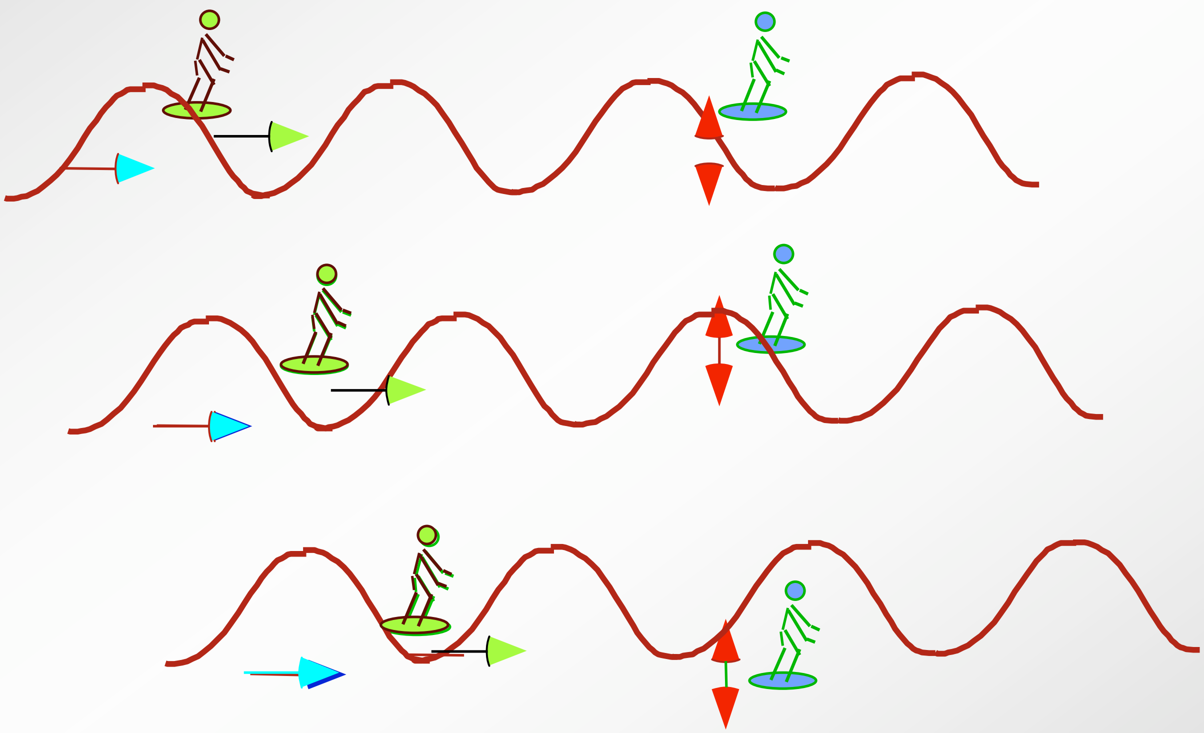
Linac literature more commonly defines "shunt impedance" without the "2"

$$\mathscr{R}_{in} = \frac{V_o^2}{\mathscr{P}} \sim \frac{1}{R_{surf}}$$

For SC-rf $\mathscr{P}$ is reduced by orders of magnitude

BUT, it is deposited @ 2K

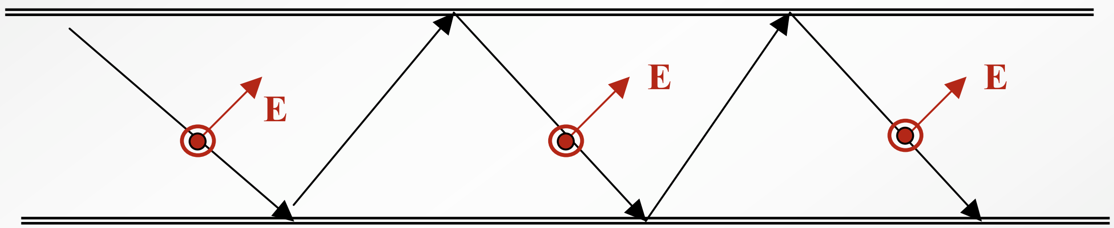
Surfing analogy of the traveling wave acceleration mechanism



To "catch" the wave the surfer must be synchronous with the phase velocity of the wave

Typically we need a longitudinal E-field to accelerate particles in vacuum

- Example: the standing wave structure in a pillbox cavity
- What about traveling waves?
  - Waves guided by perfectly conducting walls can have E_{long}



- But first, think back to phase stability
  - To get continual acceleration the wave & the particle must stay in phase
  - Therefore, we can accelerate a charge with a wave with a synchronous phase velocity, $v_{ph} \approx v_{particle} < c$

Can the accelerating structure be a simple (smooth) waveguide?

- Assume the answer is "yes"
- Then $\mathbf{E} = \mathbf{E}(r,\theta)\, e^{i(\omega t - kz)}$ with $\omega/k = v_{ph} < c$
- Transform to the frame co-moving at $v_{ph} < c$
- Then,
  - The structure is unchanged (by hypothesis)
  - E is static (v_{ph} is zero in this frame)

  ==> By Maxwell's equations, H =0

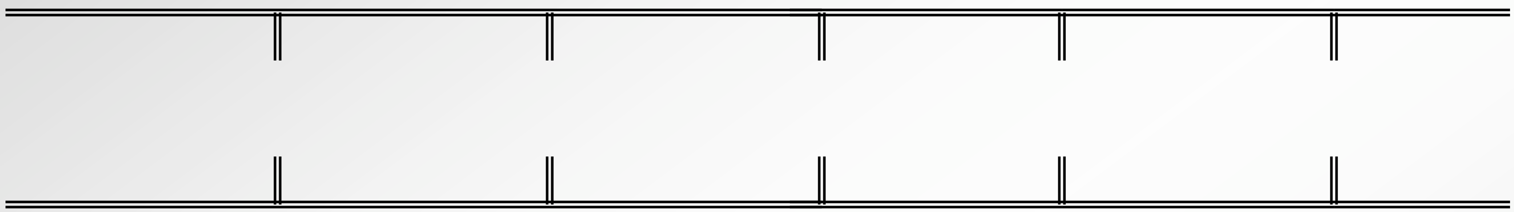
  ==> $\nabla \circ \mathbf{E} = 0$ and $\mathbf{E} = -\nabla\phi$

  - But ϕ is constant at the walls (metallic boundary conditions)

  ==> $\mathbf{E} = 0$

The assumption is false, smooth structures have $v_{ph} > c$

To slow the wave, add irises



In a transmission line the irises

a) Increase capacitance, C

b) Leave inductance ~ constant

c) ==> lower impedance, Z

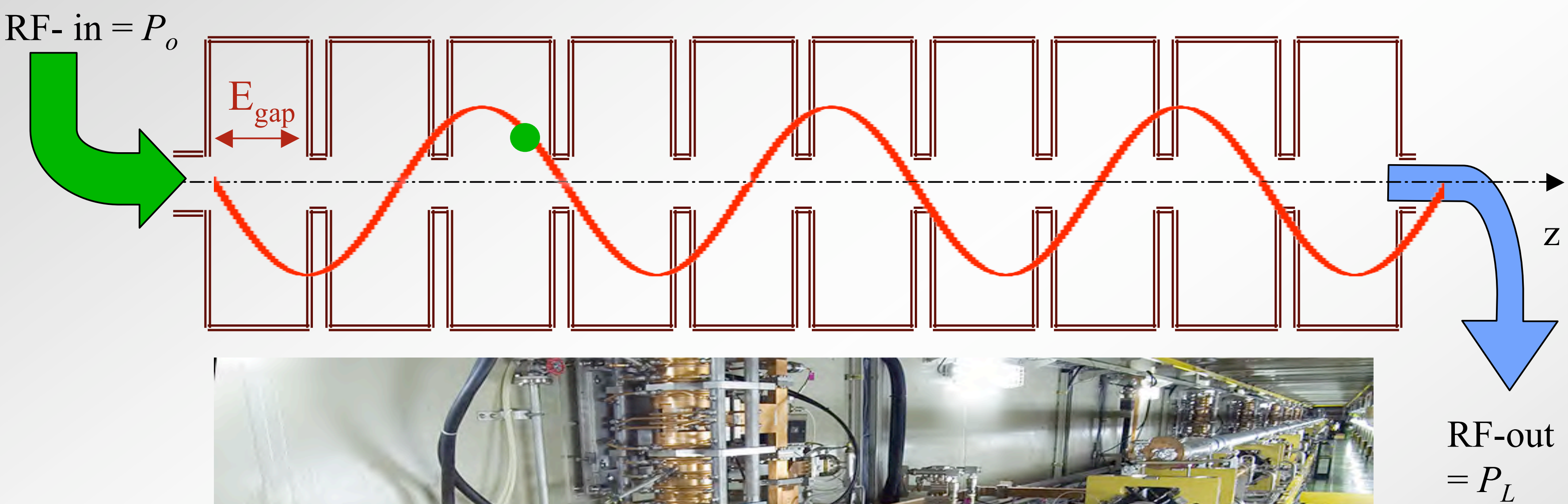
d) ==> lower v_{ph}

Similar for TM01 mode in the waveguide

$$\frac{\omega}{k} = \frac{1}{\sqrt{LC}}$$

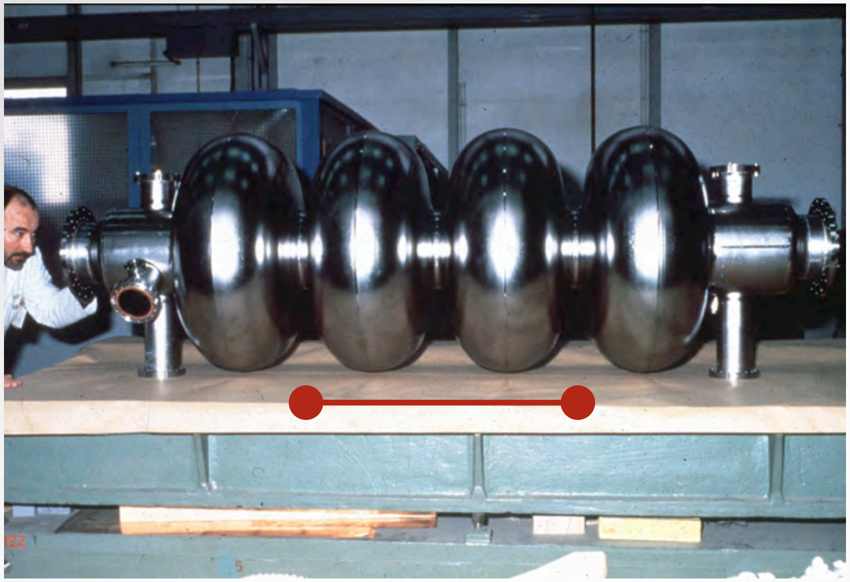
$$Z = \frac{L}{C}$$

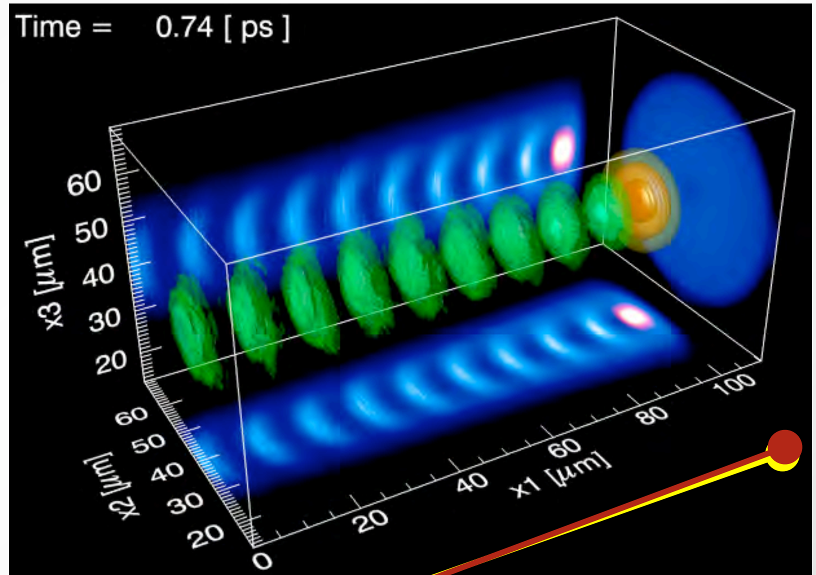
Ultra-relativistic particles (v ≈ c) can "surf" an rf field traveling at c



RF-cavities in metal and in plasma
Think back to the string of pillboxes

Courtesy of W. Mori & L. da Silva





1 m

RF cavity

100 μm

Plasma cavity

$G \sim 30\ MeV/m$

$G \sim 30\ GeV/m$

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End of unit